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An explicit solution for the elastic quarter-space problem in matrix formulation

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ABSTRACT

This paper presents a fast and convenient algorithm for the solution of the elastic quarter-space contact problem, which uses discretization to form matrices to realize the overlapping solution process for the elastic quarter-space as developed by Hetenyi [Hetenyi, M., 1970. A general solution for the elastic quarter space, Trans. ASME Journal of Applied Mechanics 37 E(1), 70–76]. This proposed method provides an explicit solution which is as yet absent in existing literatures. The generated matrices are only related to the mesh structure and Poisson's ratio while unrelated to the loading, such that they can be applied to different loading cases. Hence, the present method offers a possibility for substantially improving the efficiency of those numerical iterative analyses, such as the elastohydrodynamic lubrication of the contact in an elastic quarter-space. Verification of the present method was accomplished by comparison with the existing quarter-space results.

1. Introduction

The contact problem in an elastic quarter-space is quite common in practical mechanical systems such as the contact of rail-wheels, cam-followers, gears and roller bearings. The common characteristic of these contacts is that there are free end surfaces near the contact or loading region. Due to the difficulty of solving this problem in a purely analytical way, a number of ingenious ways were developed by various researchers. Of special significance is Hetenyi's ingenious concept of using overlapped half spaces to solve this problem (Hetenyi, 1970). A system of two coupled integral equations is formed to express the involved mathematical relations. The existing methods of solution of this system can be roughly classified into the following categories: 1. Numerical solutions performed in the physical space, which are numerical iteration (Hetenyi, 1970) and direct numerical solution (Hanson and Keer, 1990); 2. Transformation into a transformed space, followed by numerical solution, and inverse transformation of the result. Two kinds of transformation are applied, namely, Fourier transformation (Sneddon, 1971; Keer et al., 1983) and Mellin transformation (Hecker and Romanov, 1993); 3. Ritz's method based numerical solution (Guenfoud et al., 2010). The present paper strives to follow Hetenyi's and Keer's works in the aspect of solution methodology of the first category. Its main difference from the existing methods lies in the employment of matrix formulation to the discretized equations. This approach is benefited from the flexibility of matrix operations to gain, within Hetenyi's concept of reflection and overlapping, an explicit solution and characteristics unique of the elastic quarter space in the form of matrices.

As a first stage of applying the present method to attack this type of problems, the stress singularity caused by the edge loading is not considered. This does not affect its applicability to the problems without edge loads, for example the elastohydrodynamic lubrication problem which takes the boundary condition of zero pressure and excludes any non-zero boundary pressure (Zhu et al., 2009).

2. Analysis

The general loading case of the quarter space can be reduced to an equivalent case of quarter space with two loads applied respectively normal onto the top and side surfaces by overlapping of half space or half spaces (Yu et al., 1996). The case of a quarter space with normal loadings on both surfaces can further be split into two overlapped quarter spaces with one surface normally loaded and the other free. The last case can therefore be looked upon as the hard kernel of the general loading case in this sense, and its solution can be seen as a key to the solution of the general loading case. It is taken as the basic form of elastic quarter-space problem in Hetenyi's and also the present paper, as shown in Fig. 1.

Geometrically, the quarter-space can be taken as the half of a half-space model. Figs. 2 and 3 show schematically the horizontal (H) and the vertical (V) half-spaces respectively. In both of these half-spaces, the symmetrical loads on the two sides cause only normal stresses but not any shear stresses on the respective mid-sections.

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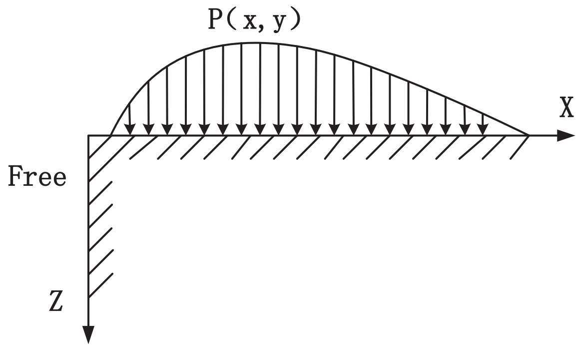


Fig. 1. Basic elastic quarter-space problem.

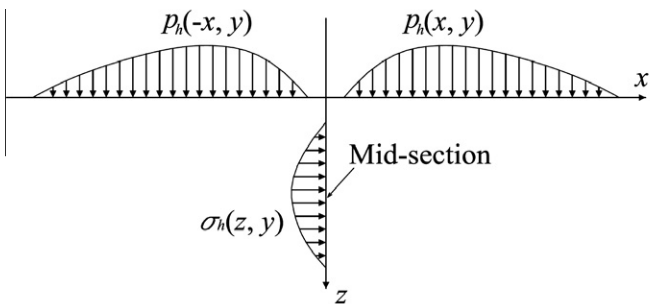


Fig. 2. Horizontal (H) half-space symmetrically loaded.

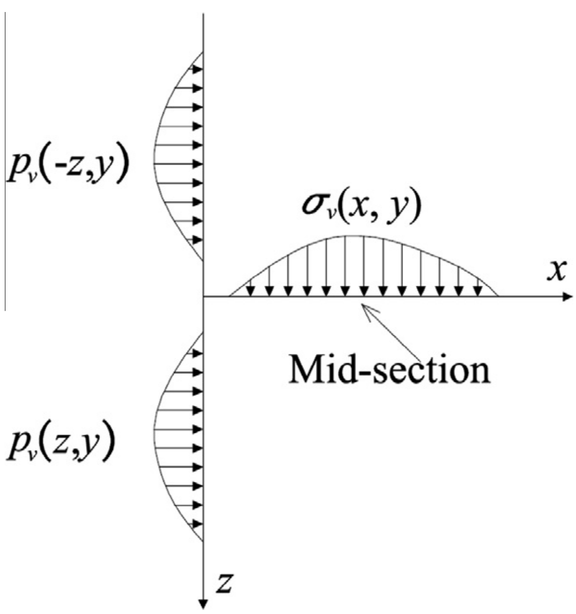


Fig. 3. Vertical (V) half-space symmetrically loaded.

The numerical treatment begins with discretization of both H- and V- surfaces. Fig. 4(a) shows the effective part of the H-surface discretized into a set of k_h rectangles. One of the rectangles, the ith rectangle, is shown in Fig. 4(b). The coordinates of its centre are denoted by (x_i,y_i), and its length and width are respectively $2\alpha_i$ and $2\beta_i$. The V-surface is discretized into a set of k_v rectangles, either geometrically similar to H- with z replacing x, or independently.

The loads on the surfaces are approached by piece-wise distributions. The values of the distributed loads on H- and V- surfaces are denoted by their values at the centers of the rectangles $(p_h)_i = p_h(x_i,y_i)$ and $(p_v)_j = p_v(z_j,y_j)$. The mid-sectional normal stresses caused by the loads with their mirror images are similarly denoted by their values at the rectangle centers $(\sigma_h)_j = \sigma_h(z_j,y_j)$ and $(\sigma_v)_i = \sigma_v(x_i,y_i)$. The arrays of these four sets of pressure and stress values form four vectors $\mathbf{P_h}$, $\mathbf{P_v}$, $\mathbf{S_h}$ and $\mathbf{S_v}$. The applied load distribution can also be approached by a piecewise distribution and expressed by a vector $\mathbf{P}$.

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix}; \quad \mathbf{P}_h = \begin{bmatrix} (p_h)_1 \\ (p_h)_2 \\ \vdots \\ (p_h)_{k_h} \end{bmatrix}; \quad \mathbf{P}_v = \begin{bmatrix} (p_v)_1 \\ (p_v)_2 \\ \vdots \\ (p_v)_{k_v} \end{bmatrix};$$

$$\mathbf{S}_h = \begin{bmatrix} (\sigma_h)_1 \\ (\sigma_h)_2 \\ \vdots \\ (\sigma_h)_{k_v} \end{bmatrix}; \quad \mathbf{S}_v = \begin{bmatrix} (\sigma_v)_1 \\ (\sigma_v)_2 \\ \vdots \\ (\sigma_v)_{k_h} \end{bmatrix}$$

The stress $\sigma_h(z_j,y_j)$ at rectangle j produced by the constant pressure $p_h(x_i,y_i)$ on the ith rectangle and its image $p_h(-x_i,y_i)$ can be expressed as:

$$\sigma_h(z_j, y_j) = m_{ij} p_h(x_i, y_i) \tag{1}$$

where the coefficient m_{ij} is determined by the Love's solution (Love, 1929). The 2-dimensional array of all the coefficients m_{ij} forms a 'reflecting' matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{21} & \cdots & m_{k_h1} \\ m_{12} & m_{22} & \cdots & m_{k_h2} \\ \vdots & & & \vdots \\ m_{1k_v} & m_{2k_v} & \cdots & m_{k_hk_v} \end{bmatrix}$$

The relation between the stress vector $\mathbf{S_h}$ and the load vector $\mathbf{P_h}$ can be expressed by the following equation:

$$\mathbf{S_h} = \mathbf{M} \cdot \mathbf{P_h} \tag{2}$$

Similarly, the vector $\mathbf{S_v}$ can also be related to the vector $\mathbf{P_v}$ by another 'reflecting' matrix $\mathbf{N}$:

$$\mathbf{S_v} = \mathbf{N} \cdot \mathbf{P_v} \tag{3}$$

The right part of H-space is overlapped on the lower part of V-space. The stresses on the boundaries of the resulting quarter space must fulfill the given conditions, and two simultaneous equations are therefore formed:

$$-\mathbf{P_h} + \mathbf{S_v} = -\mathbf{P_h} + \mathbf{N} \cdot \mathbf{P_v} = -\mathbf{P} \tag{4}$$

$$-\mathbf{P_v} + \mathbf{S_h} = -\mathbf{P_v} + \mathbf{M} \cdot \mathbf{P_h} = 0 \tag{5}$$

Eq. (5) gives $\mathbf{P_v} = \mathbf{M} \cdot \mathbf{P_h}$. Substitution of this into Eq. (4) results in a decoupled equation for $\mathbf{P_h}$:

$$\mathbf{P_h} - \mathbf{N} \cdot (\mathbf{M} \cdot \mathbf{P_h}) = \mathbf{P} \tag{6}$$

Since $\mathbf{N} \cdot (\mathbf{M} \cdot \mathbf{P_h}) = (\mathbf{N} \cdot \mathbf{M}) \cdot \mathbf{P_h}$, Eq. (6) can be written as:

$$(\mathbf{I} - \mathbf{N} \cdot \mathbf{M}) \cdot \mathbf{P_h} = \mathbf{P} \tag{7}$$

Thus, an explicit solution for the load $\mathbf{P_h}$ is readily obtained:

$$\mathbf{P_h} = \mathbf{A} \cdot \mathbf{P} \tag{8}$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{N} \cdot \mathbf{M})^{-1}$. Similarly,

$$\mathbf{P_v} = \mathbf{B} \cdot \mathbf{P} \tag{9}$$

where $\mathbf{B} = \mathbf{M} \cdot (\mathbf{I} - \mathbf{N} \cdot \mathbf{M})^{-1}$ or $\mathbf{B} = \mathbf{M} \cdot \mathbf{A}$.

With $\mathbf{P_h}$ and $\mathbf{P_v}$ known, the stress and deformation distributions within the respective half-spaces can be obtained by applying the Love's solution again, and superposed to give the stress and deformation distributions within the quarter-space.

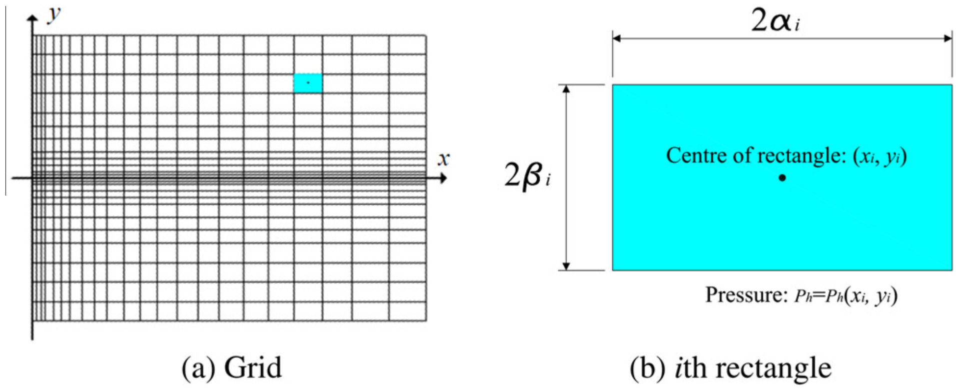


Fig. 4. Demonstration of grid meshing.

It is also interesting to note that the above solution is actually the limit of infinite Hetenyi's iteration. To show this, the matrix form is applied to Hetenyi's process. The nth iterative results are:

$$\mathbf{P}_{\mathbf{h}}^{(n)} = [\mathbf{I} + \sum_{i=1}^{n} (\mathbf{N}\cdot\mathbf{M})^{i}]\cdot\mathbf{P} \quad \text{and}$$
$$\mathbf{P}_{\mathbf{v}}^{(n)} = M\cdot[\mathbf{I} + \sum_{i=1}^{n} (\mathbf{N}\cdot\mathbf{M})^{i}]\cdot\mathbf{P} \tag{10}$$

The bounded character and convergence of such iterations were proven by Hetenyi (1970). It is evident that the limits of infinite iterations are identical to Eqs. (8) and (9).

The 'reflecting' matrices **M** and **N**, and therefore also the matrices **A** and **B**, are solely related to the mesh structure and Poisson's ratio, while independent of the loading. They are special properties unique of the elastic quarter space, extracted from the reflection and overlapping within Hetenyi's concept. Once generated, these matrices can be stored and employed in different loading cases of the same Poisson's ratio and geometrically similar mesh structure of the surfaces. This could provide substantial convenience in treating different loading cases, and particularly convenient in treating problems where repetitive calculations of the same quarter-space are necessary.

3. Case study

A case is selected from Hanson and Keer (1990), and the results obtained by the present solution are compared with those of Hanson and Keer. The very detailed loading condition and abundance of results provided by Hanson and Keer (1990) are very suitable for verification of the present method.

A half hemispherical loading is applied to the top surface of an elastic quarter-space, with the spherical centre located at the edge. The load can be described by the following formulae:

$$\begin{aligned} &p(x,y) = p_0\frac{\sqrt{a^2-x^2-y^2}}{a}, \quad \sqrt{x^2+y^2}\leqslant a \\ &p(x,y) = 0, \quad \sqrt{x^2+y^2} > a \end{aligned} \tag{11}$$

where a denotes the radius of the semi-circular region of the pressure distribution, and p_o is the maximum pressure at the centre of the semi-circle. Poisson's ratio, ν, is taken to be 0.3.The H-surface is discretized into a set of 5015 uneven-sized rectangles. In the x-direction starting from $x = 0$, the width of the first rectangle 2α is taken to be 0.01a, and the width of the succeeding rectangles is expressed in the form of a rising geometric series with a factor of 1.1 till 0.1a, and then with a factor of 1.05. In the y-direction, the first

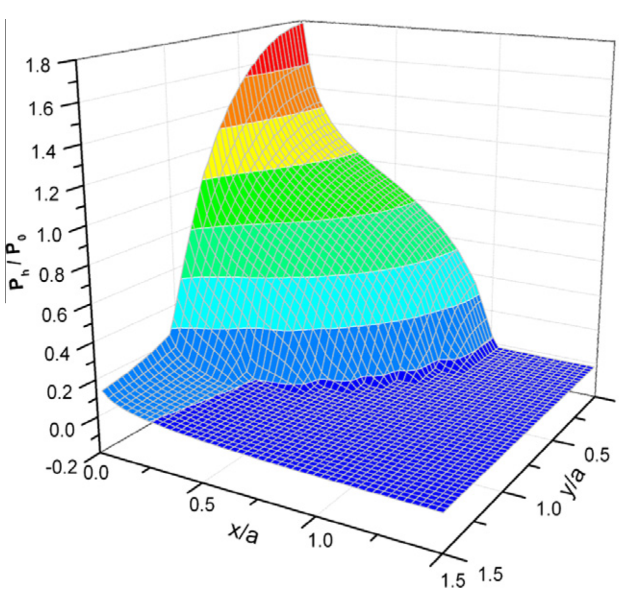


Fig. 5. Distribution of p_h on H-surface.

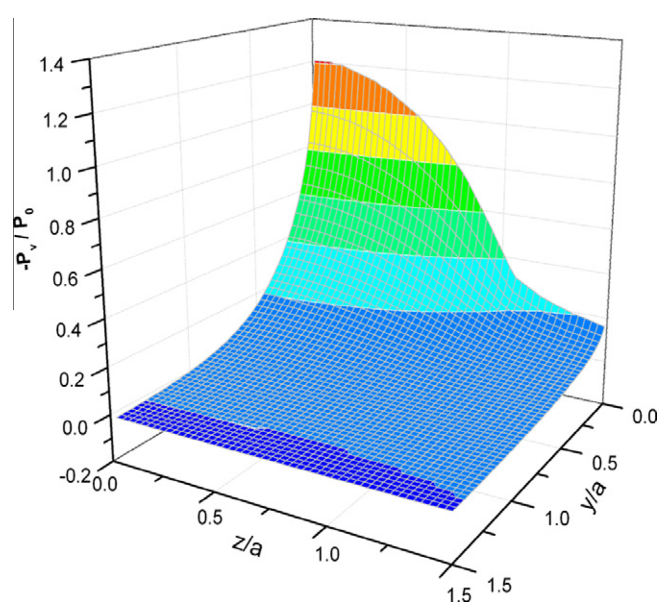
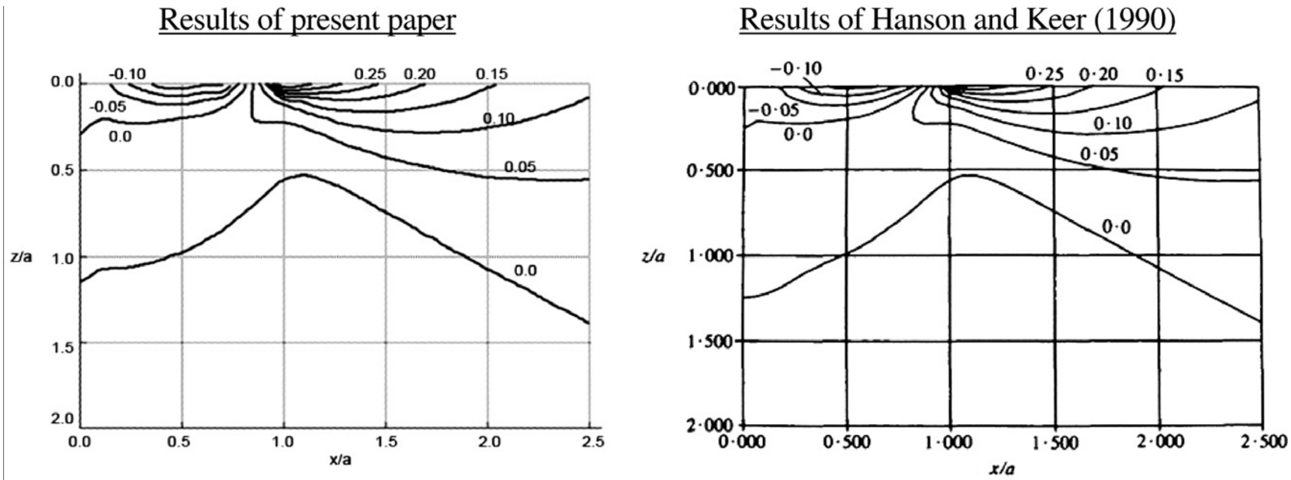
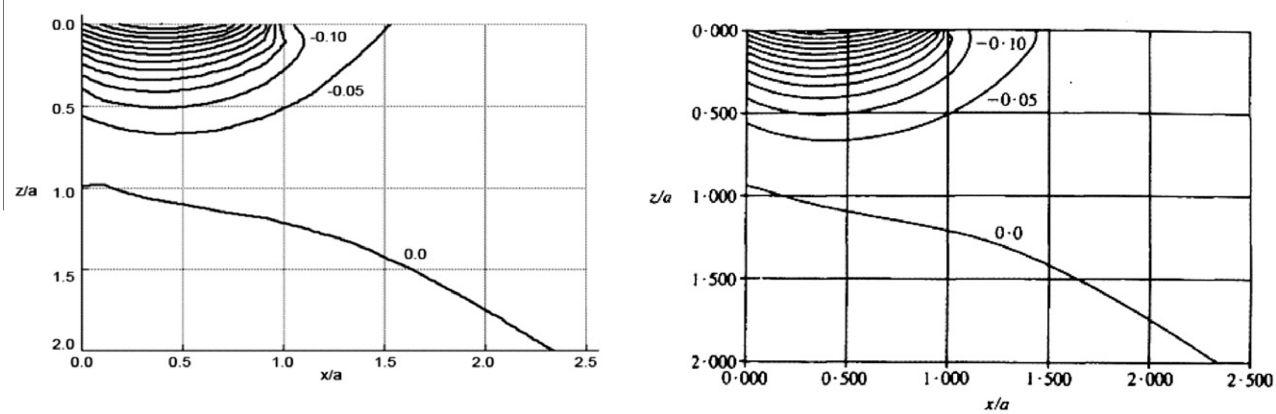


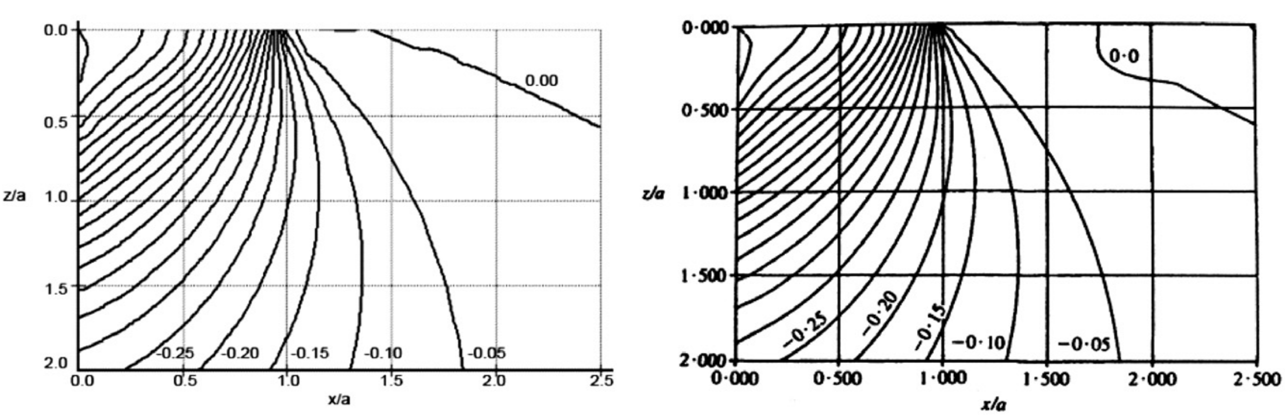
Fig. 6. Distribution of p_v on V-surface.



(a) Contours of σ_{xx}/p_0 in the (x, z)-plane



(b) Contours of σ_{yy}/p_0 in the (x, z)-plane



(c) Contours of σ_{zz}/p_0 in the (x, z)-plane

Fig. 7. Comparison of results of stress distributions obtained by present method and extracted from Hanson and Keer (1990).

rectangle centre is placed at $y = 0$ and its width 2β taken as $0.05a$. The width of the succeeding rectangles in both positive and negative y-directions initially increases with a geometric series with a factor of 1.1 and then 1.05 in the same way as above. The discretization continues until it covers a sufficiently large region with $0 \leqslant x \leqslant 9.7a$ and $-9.7a \leqslant y \leqslant 9.7a$.

The very fine meshing in the x-direction from the edge is employed to minimize the error caused by imperfectly treating the stress singularity at the edge involved in the reflecting action due to the edge loading by a distributed normal loading. It can be seen from the results that this measure is fairly effective with respect to the precision of global distributions of various stress components.The discretization of V-surface is identical to that of the H-surface with z in the place of x. Therefore, the 'reflecting' matrices $\mathbf{M}$ and $\mathbf{N}$ are identical matrices of 5015×5015, and the matrix $\mathbf{A}$ becomes $\mathbf{A} = (\mathbf{I} - \mathbf{M}^2)^{-1}$ in Eq. (8).

The required loads on H- and V-half surfaces are calculated from the load vector $\mathbf{P}$ by Eqs. (8) and (9), and the results are shown in Figs. 5 and 6.

Normal stresses within the quarter-space are calculated from the loads on H- and V-surfaces together with their images. The stress distributions on the x-z plane were obtained with the present matrix formulation and are plotted in contour together with the results of Hanson and Keer (1990) in Fig. 7. Comparing the two sets, it shows that the present results are in good correlation with Hanson and Keer's. The matrix solution is thus validated. The present solution should be conveniently applicable in practical elastic quarter-space problems which have no or insignificant effect of edge loading.

4. Conclusion

(1) The matrix formulation is employed to express the numerical equations for the elastic quarter-space, and an explicit solution is achieved, which has not been found in existing literatures. This solution is shown to be identical to the limit of Hetenyi's iteration.

(2) A special solution process is formed, wherein the involvement of the load is pushed to the last moment, after the generation of the final matrices. These matrices can be seen as characteristics unique of the elastic quarter space within Hetenyi's concept of overlapping. They can be directly applied to different given loadings. This facilitates fast and efficient calculations of the elastic stress and deformation fields of a quarter-space problem, particularly when repetitive calculations are necessary.

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